Analytic Number Theory
M. Math Midterm Exam(2022).
Maximum marks 30

General instruction: This question paper contains two parts. In question no 1. each question carries 2 marks. Question 2 to 9 carry 4 marks each. Solve any 5 out of these 8 question.

1) Answer following questions.

(a) Define multiplicative and completely multiplicative function. Is the Mangoldt function $\Lambda(n)$ multiplicative?

(b) State and prove Abel's Identity.

(c) Define Gauss sum associated with $\chi$. If $\chi$ is any Dirichlet character mod $k$, then prove that

$$G(n, \chi) = \overline{\chi(n)} G(1, \chi)$$

whenever $(n, k) = 1$.

(d) Let $f(x) = a_0 + a_1 x + \cdots a_n x^n$ be a polynomial with integer coefficients, where $a_n > 0$ and $n > 1$. Prove that $f(x)$ is composite for infinitely many integers $x$.

(e) Let $\chi$ be any real-valued character mod $k$ and let

$$A(n) = \sum_{d|n} \chi(d).$$

Then prove that $A(n) \geq 0$ for all $n$, and $A(n) \geq 1$ if $n$ is a square.

2) State and prove weak and strong versions of Dirichlet asymptotic formulae for the partial sums of the divisor function $d(n)$.

3) Let $p_n$ denote the $n$-th prime. Then prove that the following asymptotic relations are logically equivalent:

$$\lim_{x\to\infty} \frac{\pi(x) \log x}{x} = 1$$

$$\lim_{x\to\infty} \frac{\pi(x) \log(\pi(x))}{x} = 1$$

$$\lim_{x\to\infty} \frac{p_n}{n\log n} = 1$$

4) Prove that for every integer $n \geq 2$

$$\frac{n}{6\log n} \leq \pi(n)$$

5) Let a(n) be a nonnegative sequence such that

$$\sum_{n\leq n} a(n)\left[\frac{x}{n}\right] = x\log x + O(x)$$

for all $x \geq 1$
Then prove that, for $x \geq 1$

$$\sum_{n\leq x} \frac{a(n)}{n} = \log x + O(1)$$

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6) For $x \geq 1$ let

$$M(x) = \sum_{n\leq x} \mu(n).$$

Prove that $M(x) = o(x)$ as $x \to \infty$ implies Prime number theorem.

7) Prove that a finite abelian group $G$ of order $n$ has exactly $n$ distinct characters.

8) For $x > 1$ and $\chi \neq \chi_1$ prove that

$$\sum_{p\leq x} \frac{\chi(p)\log(p)}{p} = -L'(1,\chi)\sum_{n\leq x} \frac{\mu(n)\chi(n)}{n} + O(1)$$

where

$$L'(1,\chi = -\sum_{n=1}^{\infty} \frac{\chi(n)\log n}{n}.$$

9) Define conductor of a character. Prove that every Dirichlet character $\chi$ mod $k$ can be expressed as a product,

$$\chi(n) = \psi(n)\chi_1(n)$$

for all $n$, where $\chi_1$ is the principle character mod $k$ and $\psi$ is primitive character modulo the conductor of $\psi$